Kuwait University | Math 101 | Date: August 1, 2009
Dept. of Math. & Comp. Sci. | Second Exam | Duration: 90 minutes

Calculators, cellular phones and all other mobile communication equipments are not allowed

Answer the following questions:

1. Use differentials to approximate $\sqrt[3]{28}$. (3 pts.)

2. Find an equation of the normal line to the curve $\sec(x^2y) + \sqrt{x^2+y} - x = 3$ at $x = 0$. (4 pts.)

3. A snow ball is melting in such a way that its surface area is decreasing at a rate of $1\ \text{cm}^2/\text{min}$. Find the rate of change of its volume when the radius of the ball is 2 cm. (4 pts.)

4. (a) State Rolle's Theorem. (1 pt.)

   (b) Use Rolle's Theorem to show that for all $a \in \mathbb{R}$, the equation

$$\frac{3x}{2} + \sin x = a$$

   has at most one solution. (3 pts.)

5. Let $f(x) = \dfrac{4-x}{(x-2)^2}$ and given that $f'(x) = \dfrac{x-6}{(x-2)^3}$ and $f''(x) = \dfrac{2(8-x)}{(x-2)^4}$.

   (a) Find the vertical and horizontal asymptotes for the graph of f, if any.

   (b) Find the intervals on which f is increasing and the intervals on which f is decreasing. Find the local extrema of f, if any.

   (c) Find the intervals on which the graph of f is concave upward and the intervals on which the graph of f is concave downward. Find the points of inflection, if any.

   (d) Sketch the graph of f.

   (e) Find the maximum and minimum values of f on $[3, 7]$ (10 pts.)

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1. Let $f(x) = \sqrt[3]{x}$ and $x_0 = 27$, then $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$ and , $\Delta x = 1$. So, $\sqrt[3]{28} = f(28) \simeq f(27) + f'(27)\,\Delta x = 3 + \frac{1}{27}(1) = \boxed{\frac{82}{27} = 3.037}$.

2. At $x = 0$, $y = 4$. Differentiate implicitly with respect to x, we have:
$$\sec(x^2y)\tan(x^2y)(2xy + x^2y') + \frac{2x + y'}{2\sqrt{x^2 + y}} - 1 = 0$$
Therefore, $\boxed{y'|_{(0,4)} = 4} \Longrightarrow \boxed{m_{\perp} = -\frac{1}{4}}$. Equation of normal line: $\boxed{y = -\frac{1}{4}x + 4}$.

3. $S = 4\pi r^2 \Longrightarrow \frac{dS}{dt} = 8\pi r\frac{dr}{dt} \Longrightarrow -1 = 8\pi(2)\left.\frac{dr}{dt}\right|_{r=2} \Longrightarrow \left.\frac{dr}{dt}\right|_{r=2} = \boxed{\frac{-1}{16\pi}\text{ cm/min}}$
$V = \frac{4}{3}\pi r^3 \Longrightarrow \frac{dV}{dt} = 4\pi r^2\frac{dr}{dt} \Longrightarrow \left.\frac{dV}{dt}\right|_{r=2} = 4\pi(2)^2\left(\frac{-1}{16\pi}\right) = \boxed{-1\text{ cm}^3\text{/min}}$.

4. (b) Suppose x_1, x_2 are two distinct (different) solutions of the equation ($x_1 < x_2$, say). Let $f(x) = \frac{3}{2}x + \sin x - a$ and consider the interval $[x_1, x_2]$. f is continuous on $[x_1, x_2]$, f is differentiable on (x_1, x_2) and $f(x_1) = 0 = f(x_2)$ (x_1, x_2 are two roots of f). $f'(x) = \frac{3}{2} + \cos x$. From Rolle's Theorem $\exists\, c \in (x_1, x_2)$ such that $f'(c) = 0$, i.e., $\frac{3}{2} + \cos c = 0 \Longrightarrow \cos c = -\frac{3}{2}$, which is a contradiction with $-1 \leq \cos c \leq 1$.

5. Domain $f = \mathbb{R} - \{2\}$. The points $(4,0)\&(0,1)$ lie on the curve (intercepts).

   (a) $\lim_{x\to 2^{\pm}} f(x) = \infty \Longrightarrow \boxed{x = 2}$ is a *vertical asymptote*. $\lim_{x\to\pm\infty} f(x) = 0 \Longrightarrow \boxed{y = 0}$ is a *horizontal asymptote*.
   (The graph of f intersects its horizontal asymptote $y = 0$ at $x = 4$)

   (b) $f'(6) = 0$. At $x = 2$, f'does not exist (f has *infinite discontinuity*).

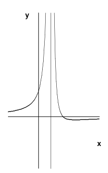
I	$(-\infty, 2)$	$(2, 6)$	$(6, \infty)$
sign of $f'(x)$	$+$	$-$	$+$
Conclusion	$\nearrow$	$\searrow$	$\nearrow$

   f is increasing on $(-\infty, 2) \cup [6, \infty)$ and f is decreasing on $(2, 6]$
   $f(6) = -\frac{1}{8}$ is a local minimum of f.

   (c) $f''(8) = 0$, and f'' does not exist at $x = 2$ where f is not continuous. (f has *infinite discontinuity*).

I	$(-\infty, 2)$	$(2, 8)$	$(8, \infty)$
sign of $f''(x)$	$+$	$+$	$-$
Concavity	*CU*	*CU*	*CD*

   , $\left(8, -\frac{1}{9}\right)$ is an inflection point.

   (d) 

   (e)

x	$f(x)$	Classification of x
3	1	end point
6	$-\frac{1}{8} = -\frac{25}{200}$	critical number
7	$-\frac{3}{25} = -\frac{24}{200}$	end point

   $f(3) = 1$ is maximum value of f on $[3, 7]$ and $f(6) = -\frac{1}{8}$ is minimum value of f on $[3, 7]$.